Double Integral of a Minimum Function

Gary Schurman MBE, CFA

December 2021

We want to solve the following double integral of a minimum function...

$$I=\int\limits_{u=s}^{u=t}\int\limits_{v=s}^{v=t} f(min(u,v))\,\delta v\,\delta u \tag{1}$$

To solve the integral in Equation (1) above we will split that integral into two parts. The first part considers cases where $v \le u$ and the second part considers cases where $u \le v$. The revised integral equation is...

$$I=\int\limits_{u=s}^{u=t}\left[\int\limits_{v=s}^{v=u} f(v)\,\delta v+\int\limits_{v=u}^{v=t} f(u)\,\delta v\right]\delta u=\int\limits_{u=s}^{u=t}\int\limits_{v=s}^{v=u} f(v)\,\delta v\,\delta u+\int\limits_{u=s}^{u=t}\int\limits_{v=u}^{v=t} f(u)\,\delta v\,\delta u \tag{2}$$

Our Hypothetical Problem

We want to solve a double integral of the following function (note that $u \wedge v$ means take the minimum of u and v)...

$$f(min(u,v))=\mathrm{Exp}\left\{-\lambda\,(u+v-2\,(u\wedge v))\right\} \tag{3}$$

Assume that we want to integrate over the time interval $[3,10]$ and that the value of λ is 0.40. The integral that we want to solve is...

$$I=\int\limits_{u=s}^{u=t}\int\limits_{v=s}^{v=u}\mathrm{Exp}\left\{-\lambda\,(u+v-2\,(u\wedge v))\right\}\delta v\,\delta u\ \text{ ...where... }\ s=3,\ t=10,\ \lambda=0.40 \tag{4}$$

Solution to the First Integral

Using Equation (2) above we will define integral I_a to be the first integral in that equation...

$$I_a=\int\limits_{u=s}^{u=t}\int\limits_{v=s}^{v=u} f(v)\,\delta v\,\delta u=\int\limits_{u=s}^{u=t}\int\limits_{v=s}^{v=u}\mathrm{Exp}\left\{-\lambda\,(u+v-2\,v)\right\}\delta v\,\delta u=\int\limits_{u=s}^{u=t}\int\limits_{v=s}^{v=u}\mathrm{Exp}\left\{-\lambda\,(u-v)\right\}\delta v\,\delta u \tag{5}$$

The solution to the inner integral in Equation (5) above is...

$$I_a\,\text{inner}=\int\limits_{v=s}^{v=u}\mathrm{Exp}\left\{-\lambda\,(u-v)\right\}\delta v=\frac{1}{\lambda}\left(\mathrm{Exp}\left\{-\lambda\,(u-u)\right\}-\mathrm{Exp}\left\{-\lambda\,(u-s)\right\}\right)=\frac{1}{\lambda}\left(1-\mathrm{Exp}\left\{-\lambda\,(u-s)\right\}\right) \tag{6}$$

Using Equation (6) above the equation for the outer integral in Equation (5) above is...

$$I_a\,\text{outer}=\int\limits_{u=s}^{u=t} I_a\,\text{inner}\,\delta u=\int\limits_{u=s}^{u=t}\frac{1}{\lambda}\left(1-\mathrm{Exp}\left\{-\lambda\,(u-s)\right\}\right)\delta u=\frac{1}{\lambda}\left[\int\limits_{u=s}^{u=t}\delta u-\int\limits_{u=s}^{u=t}\mathrm{Exp}\left\{-\lambda\,(u-s)\right\}\delta u\right] \tag{7}$$

The solution to Equation (7) above is...

$$I_a=\frac{1}{\lambda}\left[(t-s)+\frac{1}{\lambda}\left(\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}-\mathrm{Exp}\left\{-\lambda\,(s-s)\right\}\right)\right]=\frac{1}{\lambda}\left[(t-s)+\frac{1}{\lambda}\left(\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}-1\right)\right] \tag{8}$$

Solution to the Second Integral

Using Equation (2) above we will define integral I_b to be the second integral in that equation...

$$I_b = \int\limits_{u=s}^{u=t} \int\limits_{v=u}^{v=t} f(u)\,\delta v\,\delta u = \int\limits_{u=s}^{u=t} \int\limits_{v=u}^{v=t} \text{Exp}\left\{ -\lambda\,(u+v-2\,u) \right\} \delta v\,\delta u = \int\limits_{u=s}^{u=t} \int\limits_{v=u}^{v=t} \text{Exp}\left\{ -\lambda\,(v-u) \right\} \delta v\,\delta u \tag{9}$$

The solution to the inner integral in Equation (9) above is...

$$I_b\,\text{inner} = \int\limits_{v=u}^{v=t} \text{Exp}\left\{ -\lambda\,(v-u) \right\} \delta v = -\frac{1}{\lambda}\left(\text{Exp}\left\{ -\lambda\,(t-u) \right\} - \text{Exp}\left\{ -\lambda\,(u-u) \right\} \right) = \frac{1}{\lambda}\left(1 - \text{Exp}\left\{ -\lambda\,(t-u) \right\} \right) \tag{10}$$

Using Equation (10) above the equation for the outer integral in Equation (9) above is...

$$I_b\,\text{outer} = \int\limits_{u=s}^{u=t} I_b\,\text{inner}\,\delta u = \int\limits_{u=s}^{u=t} \frac{1}{\lambda}\left(1 - \text{Exp}\left\{ -\lambda\,(t-u) \right\} \right) \delta u = \frac{1}{\lambda}\left[\int\limits_{u=s}^{u=t} \delta u - \int\limits_{u=s}^{u=t} \text{Exp}\left\{ -\lambda\,(t-u) \right\} \delta u \right] \tag{11}$$

The solution to Equation (11) above is...

$$I_b = \frac{1}{\lambda}\left[(t-s) - \frac{1}{\lambda}\left(\text{Exp}\left\{ -\lambda\,(t-t) \right\} - \text{Exp}\left\{ -\lambda\,(t-s) \right\} \right) \right] = \frac{1}{\lambda}\left[(t-s) + \frac{1}{\lambda}\left(\text{Exp}\left\{ -\lambda\,(t-s) \right\} - 1 \right) \right] \tag{12}$$

Solution To Our Hypothetical Problem

Using Equations (8) and (12) above the equation for our double integral of our minimum function (Equation (3)) is...

$$I = \frac{2}{\lambda}\left[(t-s) + \frac{1}{\lambda}\left(\text{Exp}\left\{ -\lambda\,(t-s) \right\} - 1 \right) \right] \tag{13}$$

Using the parameters to our problem and Equation (13) above the answer to our problem is...

$$I = \frac{2}{0.40} \times \left[(10-3) + \frac{1}{0.40} \times \left(\text{Exp}\left\{ -0.40 \times (10-3) \right\} - 1 \right) \right] = 23.26 \tag{14}$$